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S. Barbarossa
Sapienza University of Rome

Anna Scaglione
Arizona State University

ADAPTIVE DETECTION OF POLYNOMIAL-PHASE SIGNALS EMBEDDED IN NOISE USING HIGH-ORDER AMBIGUITY FUNCTIONS

S.Barbarossa, R. Mameli, A. Scaglione,

Infocom Dept., Univ. of Rome "La Sapienza", Via Eudossiana 18, 00184 Roma, ITALY

{sergio, annas}@infocom.ing.uniroma1.it, Fax: (+39) 6-4873300

Abstract

The parameter estimation of polynomial-phase signals (PPS) has been extensively studied in the recent literature in view of possible important applications in remote sensing as well as in telecommunications. However, the detection of PPS has not received similar attention. In this paper we propose and analyze an adaptive method for the detection of PPS embedded in white Gaussian noise based on the use of the so called product high order ambiguity function.

1 Introduction

The parameter estimation of polynomial-phase signals (PPS) has received considerable attention in the literature, e.g. see [7] and the references therein, but, rather surprisingly, no comparable attention has been addressed to the detection problem. In view of possible interesting applications of the PPS modeling to radar remote sensing, where the phase modulation induced by the relative radar-target motion can be well approximated by a low order polynomial, the detection problem has to be properly analyzed. The aim of this contribution is to try to fill the gap in the analysis of the detection of PPS. More specifically, we will propose an *adaptive* detection scheme and will study its performance in the presence of additive white Gaussian noise (AWGN). The use of an adaptive threshold is motivated as follows. The basic tool for the analysis of PPS's used in this paper is the so-called Product-High order Ambiguity Function (PHAF), proposed in [2] as a way to improve the performance of the high order ambiguity function (HAF), introduced and analyzed in [6], in the presence of noise and of multiple component PPS's. The detection scheme proposed in this paper consists in the application of the Neyman-Pearson criterion, usually employed in radar applications, to the samples of the PHAF (HAF), in the frequency domain. This is indeed a generalization of the conventional theory used for the detection of sinusoids, i.e. PPS's of degree 1, embedded in white noise. In that case, the optimum detector applies a linear transformation, e.g. the Fourier Transform (FT), to the received sequence and thresholds the data in the frequency domain. That is a particular case of our method, corresponding to a polynomial order $M = 1$, because the HAF (PHAF) of order $M = 1$ coincides with the Fourier Transform. For higher order polynomials, we apply the same decision criterion to the samples of the PHAF, but in such cases, the PHAF (HAF) is nonlinear. As a consequence, some interesting peculiarities arise because of the nonlinearity. More specifically, the Neyman-Pearson criterion aims to maximize the probability of detection P_D, for a fixed probability of false alarm P_{fa}; the decision threshold is set up in order to provide the desired P_{fa} in the presence of noise only. However, if the received signal undergoes to a nonlinear transformation, the threshold evaluated considering the noise only is clearly an underestimate, because it does not take into account the interactions between signal and noise induced by the nonlinearity. Indeed, the useful signal occupies only one sample in the frequency domain, but the cross terms due to the signal-noise interactions may occupy several samples, thus yielding several false alarms. For this reason, we need to use an adaptive threshold which keeps the P_{fa} as independent as possible of noise and cross terms. To set up a proper threshold and insure the so called constant false alarm rate (CFAR) condition, we need to know the probability density function of the PHAF (HAF), in the presence of noise and signal plus noise. As a consequence, the paper is organized as follows: Section 2, after briefly reviewing the definition of PHAF, presents a statistical analysis of the PHAF in the presence of white Gaussian noise and of signal plus AWGN; in Section 3 we propose an adaptive threshold scheme providing constant false alarm rates; Section 4 shows the detection curves.

2 Statistical analysis of the PHAF

We initially recall the definition of the PHAF and then proceed to analyze its statistical properties.

2.1 Definition of HAF and PHAF

Given a finite length sequence $s(n)$, defined in the interval $|n| \leq (N-1)/2$, its M-th order multilag high order instantaneous moment (ml-HIM) is given by the application of the following recursive rule:

$$\begin{aligned} s_1(n) &:= s(n), \\ s_2(n;\tau_1) &:= s_1(n+\tau_1)s_1^*(n-\tau_1), \\ &\ldots, \\ s_M(n;\boldsymbol{\tau}_{M-1}) &:= s_{M-1}(n+\tau_{M-1};\boldsymbol{\tau}_{M-1}) \\ &\cdot s_{M-1}^*(n-\tau_{M-1};\boldsymbol{\tau}_{M-1}), \end{aligned} \quad (1)$$

where $\boldsymbol{\tau}_{M-1} := (\tau_1, \tau_2, \ldots, \tau_{M-1})$ is the vector containing all the lags used in (1). The lags are all assumed to be strictly positive. We define the multilag (ml) high-order ambiguity function (HAF) as the discrete Fourier transform of the ml-HIM:

$$S_M(f;\boldsymbol{\tau}_{M-1}) := \sum_{n=-(N-1)/2}^{(N-1)/2} s_M(n;\boldsymbol{\tau}_{M-1})e^{-j2\pi fn} \,. \quad (2)$$

The High-order Ambiguity Function (HAF) [7] is a special case of the ml-HAF, corresponding to the situation in which the lags are all equal to each other, i.e. $\boldsymbol{\tau}_{M-1} = (\tau, \tau, \ldots, \tau)$. The HAF can also be interpreted as a sample estimator of the Mth order cyclic moment [4]. Indeed, an Mth order PPS can be viewed as an Mth order cyclostationary process, i.e. a process whose Mth order cyclic moment is a stationary process. This relationship is important because it allows us to exploit all the asymptotic properties derived in [4] about the sample cyclic moments of Mth order cyclostationary processes.

The basic property of the HIM, which justifies its definition, is that the M-th order ml-HIM of a PPS signal of degree M, i.e.:

$$s(n) = Ae^{j2\pi \sum_{m=0}^{M} a_m n^m}; \qquad |n| \leq (N-1)/2 \quad (3)$$

obtained using the set of lags $\boldsymbol{\tau}_{M-1}^{(l)} = (\tau_1^{(l)}, \tau_2^{(l)}, \ldots, \tau_{M-1}^{(l)})$, is a sinusoid (see [2]):

$$s_M(n;\boldsymbol{\tau}_{M-1}^{(l)}) = A^{2^{M-1}} e^{j2\pi 2^{M-1} M! \prod_{k=1}^{M-1} \tau_k^{(l)}(a_{M-1}+a_M n)}, \quad (4)$$

assuming significant values in the interval $|n| \leq (N-1)/2 - \sum_{k=1}^{M-1} \tau_k^{(l)}$. Therefore the corresponding HAF has a peak at $f = f_0 = 2^{M-1}M! \prod_{k=1}^{M-1} \tau_k^{(l)} a_M$ [2]. This means that the estimation $\hat{a}_M$ of the highest order coefficient of an M-th degree PPS can be obtained by searching for the peak of its M-th order HAF [7]. The estimation of a_M allows us to remove the M-th order contribution from the phase of the observed signal by multiplication with the reference signal $e^{-j2\pi a_M n^M}$. If the estimation is correct, the resulting signal is an $(M-1)$-th order PPS, whose highest order coefficient can be estimated using the $(M-1)$-th order HAF and so on, up to the first order phase coefficient. This is the basic idea underlying the HAF. However, as shown in [2], the HAF presents spurious peaks when the input signal is given by the sum of PPS's having the same highest order coefficients. The ambiguities can be strongly attenuated using the PHAF, computed multiplying the ML-HAFs obtained using L different sets of lags, after proper rescaling [2]:

$$S_M^L(f;\boldsymbol{T}_{M-1}^L) = \prod_{l=1}^{L} S_M\left(\frac{\prod_{k=1}^{M-1}\tau_k^{(l)}}{\prod_{k=1}^{M-1}\tau_k^{(1)}} f; \boldsymbol{\tau}_{M-1}^{(l)}\right), \quad (5)$$

where $\tau_k^{(l)}$ indicates the k-th component of the l-th set and $\boldsymbol{T}_{M-1}^L$ is the matrix containing all the sets of lags $\boldsymbol{\tau}_{M-1}^{(1)}, \boldsymbol{\tau}_{M-1}^{(2)}, .., \boldsymbol{\tau}_{M-1}^{(L)}$. The main property of the PHAF is that, after rescaling, the useful peaks remain in the same positions, whereas the spurious peaks move along the frequency axis, so that the multiplication strongly enhances the useful peaks with respect to the spurious ones [2].

2.2 Analysis of the HAF in the presence of noise

We consider here the HAF of a sequence of independent identically distributed (iid) complex Gaussian random variables (rv) $w(n)$. Using the results derived in [4] about the sample cyclic moments, and considering that a PPS embedded in a stationary AWGN respects the mixing conditions required in [4], we may directly conclude that the samples of the HAF are asymptotically normal. The first order statistics thus depend only on the expected value and of the variance. The evaluation of the variance for a generic order M is not an easy task, because of the many factors involved in the definition of HAF. However, for low orders we can compute expected value and variance by simple substitution. In particular, for $M = 2$ we have

$$\begin{aligned} E\{W_2(f;\tau_1)\} &= E\{\textstyle\sum_n w(n+\tau_1)w^*(n-\tau_1)e^{-j2\pi nf}\} \\ &= \textstyle\sum_n R_W(2\tau_1)e^{-j2\pi nf} = 0, \end{aligned} \quad (6)$$

where $R_W(n)$ indicates the covariance of $w(n)$ and the noise whiteness assumption, i.e. $R_W(n) = \sigma_w^2\delta(n)$, has been exploited. Similarly, for $M = 3$ we have:

$$\begin{aligned} &E\{W_3(f;\tau_1,\tau_2)\} = \\ &\textstyle\sum_n (|R_W(2\tau_1)|^2 + |R_W(2\tau_2)|^2)e^{-j2\pi nf} = 0. \end{aligned} \quad (7)$$

As far as the variances are concerned, we obtain:

$$var\{W_2(f;\tau_1)\} = \sigma_w^4(N-2\tau_1), \quad (8)$$

and

$$var\{W_3(f;\tau_1,\tau_2)\} = \sigma_w^8(N-2(\tau_1+\tau_2))(1+\delta(\tau_1-\tau_2)). \quad (9)$$

It is interesting to notice that the variance is independent of the frequency f and, for $M=3$, the use of the same lag, i.e. $\tau_1=\tau_2$, leads to a higher variance.

2.3 Analysis of the HAF in the presence of signal plus noise

We consider now a signal composed by the sum of a PPS plus AWGN, i.e.

$$x(n) = Ae^{j2\pi\sum_{m=0}^{M} a_m n^m} + w(n). \quad (10)$$

Again, using the analysis of [4] applied to the sum of a deterministic component plus a stationary process, we may state that the HAF is asymptotically normal. In this case, however, the computation of the moments of the HAF is more difficult because of the cross products between signal and noise. Nevertheless, it is possible to derive an approximate expression, valid for high SNR, using a small perturbation approach. The general expression, valid for any order M is not reported for lack of space (see [3]). In the simple case of $M=2$ and $M=3$, we have:

$$var\{W_2(f;\tau_1)\} = 2(N-2\tau_1)\frac{A^4}{SNR}u_{-1}(N-2\tau_1) \quad (11)$$

and

$$\begin{aligned} var\{W_3(f;\tau_1,\tau_2)\} &= \tfrac{2A^8}{SNR} \\ \cdot\{2[N-2(\tau_1+\tau_2)]&u_{-1}(N-2(\tau_1+\tau_2)) \\ +(N-4max(\tau_1,\tau_2))&cos(4\pi(f-f_0)|\tau_1-\tau_2|) \\ \cdot u_{-1}&(N-4max(\tau_1,\tau_2)) \\ +(N-4(\tau_1+\tau_2))&cos(4\pi(f-f_0)(\tau_1+\tau_2)) \\ \cdot u_{-1}&(N-4(\tau_1+\tau_2))\} \end{aligned} \quad (12)$$

where $f_0 = 2^{M-1}M!\prod_{k=1}^{M-1}\tau_k a_M$, and $u_{-1}(n)$ denotes the unitary step function. It is interesting to observe that the variance, for $M=3$, depends on the frequency.

Example

Let us consider the sum of a PPS of order $M=3$ plus AWGN, with the following parameters: $N=192$, $A=1$, SNR=18 dB, $\tau_1=34$, $\tau_2=30$. In such a case, the theoretical variance estimated using the small perturbation approach provides the following expression:

$$var\{W_3(f;\tau_1,\tau_2)\} = 4+1.75cos(16\pi(f-f_0)). \quad (13)$$

In Fig.1 we report the theoretical expression together with the estimation made over 4000 independent Montecarlo trials. We can observe a very good agreement between simulations and theoretical results.

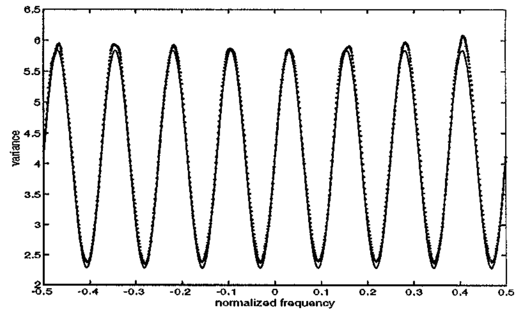

Figure 1. Variance of the HAF vs. frequency ($M=3$, SNR=18 dB) - theoretical results (solid line), simulations (dotted line).

2.4 Statistical analysis of the PHAF

It is rather difficult to derive a closed form expression for the pdf of the PHAF for a generic value of L. However, for $L=2$ it is still possible to find a closed form expression for the pdf of the square modulus of the PHAF. Recalling from previous section that the samples of the HAF are complex Gaussian random variables, we can immediately state that their square modulus has an exponential distribution. Therefore, the samples of the square modulus of the PHAF, for $L=2$, are random variables given by the product of exponential rv's. Depending on the presence of the signal and on the order M, the variance of the samples of the HAF may have different values as a function of the frequency f. Denoting by $\sigma_M^2(f,\boldsymbol{\tau})$ the variance of the HAF computed using the lags contained in the vector $\boldsymbol{\tau}$, at order M, as a function of the frequency f, the square modulus of the generic ith sample X_i of the HAF has a pdf:

$$p_X(x;f,\boldsymbol{\tau}) = \frac{1}{\sigma_M^2(f,\boldsymbol{\tau})}e^{-x/\sigma_M^2(f,\boldsymbol{\tau})}u_{-1}(x). \quad (14)$$

Therefore, taking the product of HAF's evaluated using two generic sets of lags $\boldsymbol{\tau}'$ and $\boldsymbol{\tau}''$, we obtain a random variable whose square modulus Y is the product of two exponential rv's. Then its pdf can be evaluated as follows [5]:

$$p_Y(y;f,\boldsymbol{\tau}',\boldsymbol{\tau}'') = \int_{-\infty}^{\infty}\frac{1}{|x|}p_{X_1,X_2}(x,y/x;f,\boldsymbol{\tau}',\boldsymbol{\tau}'')dx. \quad (15)$$

Indeed, the samples of HAF's corresponding to different sets of lags are independent [3]. Therefore, using (14), we obtain:

$$\begin{aligned} p_Y(y;f,\boldsymbol{\tau}',\boldsymbol{\tau}'') &= \tfrac{1}{\sigma_M^2(f,\boldsymbol{\tau}')\sigma_M^2(f,\boldsymbol{\tau}')} \\ \int_0^{\infty}\tfrac{1}{x_1}&e^{-(x_1/\sigma_M^2(f,\boldsymbol{\tau}')+y/x_1\sigma_M^2(f,\boldsymbol{\tau}''))}dx_1 \end{aligned} \quad (16)$$

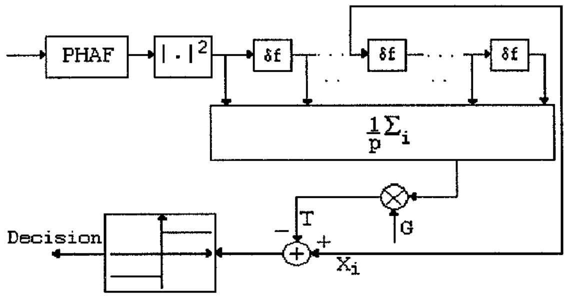

Figure 2. Decision scheme.

and solving the integral, we can express the pdf of the product in a closed form:

$$p_Y(y; f, \tau', \tau'') = \frac{2}{\sigma_M^2(f,\tau')\sigma_M^2(f,\tau'')} \cdot K_0\left(\frac{2y}{\sigma_M(f,\tau')\sigma_M(f,\tau'')}\right)u_{-1}(y), \quad (17)$$

where $K_0(u)$ is the modified Bessel function of order 0 [1]. It is worth noticing that the function $K_0(y)$ exhibits a singularity in $y = 0$, testifying a greater probability of having low values of the PHAF ($L > 1$), with respect to the HAF ($L = 1$).

3 Adaptive detection

Since the variance in (14) is a function of SNR, order and frequency (see eq. (12) and Fig. 1), it is evident that the detection of PPS's must be carried out in a adaptive way. In particular, the decision threshold must be computed adaptively as a function of the observed sequence. In this paper we propose the detection scheme depicted in Fig.2. In such a scheme, the center sample of the tapped delay line (the delay is δf) is compared with the average of the neighbour samples, multiplied by a proper gain factor G, used to induce the desired false alarm probability P_{fa}. More specifically, denoting by X_i the ith sample of the square modulus of the Mth order HAF (PHAF), the decision strategy consists in comparing X_i with a threshold T computed as the sample average of P samples located symmetrically around X_i:

$$T = \frac{G}{P} \sum_{\substack{k=-P/2 \\ k \neq 0}}^{P/2} X_{i+k} \quad (18)$$

If the central sample exceeds the threshold, we detect the presence of an Mth order PPS.

The rationale underlying the structure depicted in Fig.2 is the following. The presence of an Mth order PPS produces an isolated peak in the Mth order PHAF. Thus, if a sample of the PHAF is considerably higher than its neighbours or, according to the scheme of Fig. 2, than the average value of its neighbours, we decide for the presence of a PPS. Conversely, if a sample is not very different from its adjacent samples, we decide for the absence of a PPS. If the noise increases, for any reason, thus potentially leading to an increase of the P_{fa}, the threshold increases thus causing a dicrease of the 2. Based on the statistical analysis of sections 2.2,2.3 and 2.4 we now can prove the following result:

Result: *For $N \to \infty$, the decision criterion illustrated in Fig.2 has the following properties: a) constant P_{fa} in the presence of noise only; b) constant P_{fa}, in the presence of signal plus AWGN, for $M = 2$; c) weak variation of the P_{fa}, as a function of the SNR, in the presence of signal plus AWGN, for $M > 2$.*

In other words the scheme of Fig.2, guarantees the so called constant false alarm rate (CFAR) condition for $N \to \infty$, in the cases a) and b) for any SNR, and the in case c) for high SNR.

Proof: For N going to infinity, the samples X_i at the input of the tapped delay line of Fig.2 are exponentially distributed. IN the presence of noise only, or signal plus noise, with $M \leq 2$ (e.g. cases a) and b)), the rv's X_i all have the same variance $\sigma_i^2 = 1/\lambda$, and the threshold T is a gamma-distributed random variable [5] with pdf:

$$p_T(t) = \frac{1}{(G\sigma_M^2)^P \Gamma(P)} t^{P-1} e^{-t/(G\sigma_M^2)} u_{-1}(t). \quad (19)$$

The false alarm probability is:

$$P_{fa} = Prob\{X_n > T\} = \int_{-\infty}^{\infty} p_T(t)(1 - D_{X_n}(t))dt. \quad (20)$$

where $$D_{X_i}(t) = (1 - e^{-t/\sigma_M^2})u_{-1}(t) \quad (21)$$

is the cumulative distribution function of X_i. Substituting (19) and (21) in (20), we obtain

$$P_{fa} = \int_0^{\infty} \frac{1}{(G\sigma_M^2)^P \Gamma(P)} t^{P-1} e^{-t(G+1)/(G\sigma_M^2)} dt = \frac{1}{(G+1)^P}. \quad (22)$$

This proves parts a) and b) of the theorem. In case c), the variances of the rv's X_{i+k} assume different values $\sigma^2_{M_{i+k}}$, but the P_{fa} can still be expressed in closed form as:

$$P_{fa} = \frac{1}{\prod_{i=1}^{P}(1 + G\frac{\sigma^2_{M_{i+k}}}{\sigma^2_{M_n}})}. \quad (23)$$

In this case, the independence of the P_{fa} from the SNR cannot be guaranteed exactly. However, the P_{fa} still depends only on the ratios between the variances, which makes the dependence of the P_{fa} from the SNR an infinitesimal of order greater than $1/SNR$ [3].

The case $L > 2$ is certainly more difficult to analyze because it is not simple to derive closed form expressions for the P_{fa}. Therefore we have evaluated by simulation the gain G necessary to obtain a desired P_{fa}.

The results are shown in Fig.3, where the values of G estimated by simulation are reported as a function the P_{fa}, for orders L=1, 2 and 3 (solid lines). The dashed line shows the theoretical value of G, for $L = 1$.

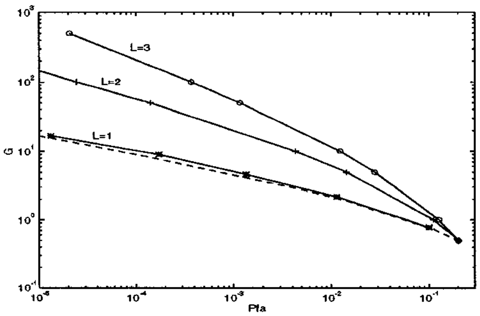

Figure 3. Gain factor G vs. P_{fa} .

4 Performance and concluding remarks

Using the results of previous section, we are now able to set up a detection scheme able to guarantee the CFAR condition. Using the factor G necessary to achieve a prescribed P_{fa}, we are able to evaluate the detection probability (P_D) as a function of the signal-to-noise ratio (SNR). The simulation results are shown in Fig.4 for the cases L= 1 and 2, obtained for three values of the P_{fa}: 10^{-4}, 10^{-3} and 10^{-2}). The ordinate axis reports the $erf^{-1}(P_D)$, where $erf^{-1}(x)$ indicates the inverse error function of x. This way of representing the P_D is rather common in radar applications because it makes more evident the values around 1 of the P_D (e.g., see [8]). The corresponding estimated P_{fa} are reported in Fig.5, where we can see that the method exhibits a good CFAR behavior, even at low SNR. From Fig.4 is clear the gain in SNR obtained using the PHAF (i.e. $L > 1$) with respect to the HAF (i.e. $L = 1$). Evaluating the gain as the decrease of SNR necessary to achieve the same P_D, for a given P_{fa}, we observe that, simply passing from $L = 1$ to $L = 2$, we have a gain of the order of about 3 dB.

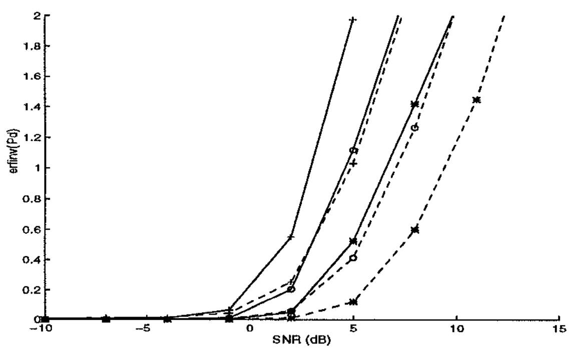

Figure 4. $Erf^{-1}(P_D)$ vs. SNR (dB) for $L = 1$ (dashed line) and $L = 2$ (solid line) with Pfa=1.e-4 ('*'), Pfa=1.e-3 ('o'), and Pfa=1.e-2 ('+').

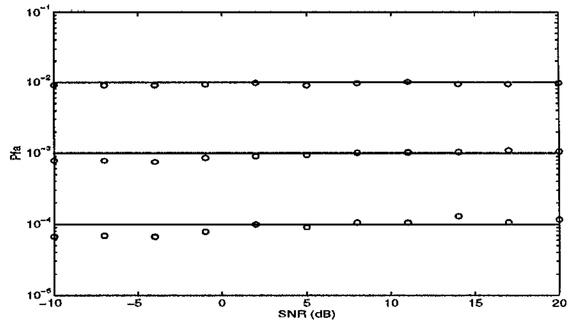

Figure 5. Estimated P_{fa}: desired values (solid line) and simulation results (circles).

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